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CUBICALLY ANISOTROPIC STRESS-STRAIN STATES AND FINDING THEM BY THE METHOD OF SMALL PARAMETER

I. M. Martynenko,[a] **M. A. Zhuravkov,**[a]
V. A. Kazakevich,[b] **and O. N. Sklyar**[b]

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The application of the method of small parameter to finding the stress-strain state of cubically anisotropic bodies in three-dimensional spatial domains in the case of rectilinear elastic anisotropy has been given; the systems of differential equations of second order for its solution have been derived.

Keywords: *cubically anisotropic body, stressed-strained state, method of small parameter.*

Introduction. The method of small parameter plays an important role in applied mathematics. Its use shows that when nonlinear or linear problems are solved, we need only restrict ourselves to two or three terms of the asymptotic series, which makes the mathematical difficulties of integration of the corresponding problems less severe without diminishing the exactness of the resulting solution [1]. Therefore, this method is widely used in elasticity and thermoelasticity theories and others. In the present work, we develop the method of small parameter as applied to the investigation of static problems of cubically anisotropic bodies.

Formulation of the Problem. The processes of straining of cubically anisotropic bodies are described by Hooke's law with three material constants. A temperature term is added to them in Hooke's law in a thermoelastic formulation. Below, we apply the method of small parameter to solving static problems of cubically anisotropic bodies in the spatial case, which is described by the following system of equations:

$$\sigma_{ii} = (A_{11} - A_{12})\,\varepsilon_{ii} + A_{12}\theta\,, \quad \sigma_{ij} = 2A_{44}\varepsilon_{ij}\,, \quad \sum_{\alpha=1}^{3} \partial_\alpha \sigma_{\alpha\beta} = 0\,, \quad i \neq j = 1, 2, 3\,, \quad \beta = \overline{1, 3}\,,$$

or

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{44}\frac{\partial^2 u}{\partial y^2} + A_{44}\frac{\partial^2 u}{\partial z^2} + (A_{12} + A_{44})\frac{\partial^2 v}{\partial x \partial y} + (A_{12} + A_{44})\frac{\partial^2 w}{\partial x \partial z} = 0\,,$$

$$A_{44}\frac{\partial^2 v}{\partial x^2} + A_{11}\frac{\partial^2 v}{\partial y^2} + A_{44}\frac{\partial^2 v}{\partial z^2} + (A_{12} + A_{44})\frac{\partial^2 u}{\partial x \partial y} + (A_{12} + A_{44})\frac{\partial^2 w}{\partial x \partial z} = 0\,, \qquad (1)$$

$$A_{44}\frac{\partial^2 w}{\partial x^2} + A_{44}\frac{\partial^2 w}{\partial y^2} + A_{11}\frac{\partial^2 w}{\partial z^2} + (A_{12} + A_{44})\frac{\partial^2 v}{\partial y \partial z} + (A_{12} + A_{44})\frac{\partial^2 u}{\partial x \partial z} = 0\,.$$

System (1) can be transformed to the following form:

$$\frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial^2 u}{\partial y^2} + \varepsilon\frac{\partial^2 u}{\partial z^2} + e\frac{\partial^2 v}{\partial x \partial y} + e\frac{\partial^2 w}{\partial x \partial z} = 0\,,$$

[a]Belarusian State University, 4 Nezavisimost' Ave., Minsk, 220050, Belarus; [b]Belarusian National Technical University, Minsk, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 82, No. 3, pp. 586–589, May–June, 2009. Original article submitted March 19, 2008.

$$\varepsilon \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \varepsilon \frac{\partial^2 v}{\partial z^2} + e \frac{\partial^2 u}{\partial x \partial y} + e \frac{\partial^2 w}{\partial x \partial z} = 0 , \tag{2}$$

$$\varepsilon \frac{\partial^2 w}{\partial x^2} + \varepsilon \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + e \frac{\partial^2 v}{\partial y \partial z} + e \frac{\partial^2 u}{\partial x \partial z} = 0 .$$

Here we have $\varepsilon = (A_{44}/A_{11})$ and $e = (A_{12} + A_{44})/A_{11}$. If A_{44} is less than A_{11} and A_{12}, the ratio $\varepsilon = (A_{44}/A_{11}) < 1$ may be considered as a small parameter. We introduce the following transformation of the coordinates and the sought functions:

$$\xi_1 = \varepsilon^{1/2} x , \quad \eta_1 = y , \quad \zeta_1 = z , \quad u = U^{(1)} , \quad v = \varepsilon^{3/2} V^{(1)} , \quad w = \varepsilon^{3/2} W^{(1)} , \tag{3}$$

$$\xi_2 = x , \quad \eta_2 = \varepsilon^{1/2} y , \quad \zeta_2 = z , \quad u = \varepsilon^{3/2} U^{(2)} , \quad v = V^{(2)} , \quad w = \varepsilon^{3/2} W^{(2)} , \tag{4}$$

$$\xi_3 = x , \quad \eta_3 = y , \quad \zeta_3 = \varepsilon^{1/2} z , \quad u = \varepsilon^{3/2} U^{(3)} , \quad v = \varepsilon^{3/2} V^{(3)} , \quad w = W^{(3)} . \tag{5}$$

We represent the components of the displacement vector as a superposition of three solutions of the form $u = u_1 + u_2 + u_3$, $v = v_1 + v_2 + v_3$, and $w = w_1 + w_2 + w_3$ which in turn are expressed by $U^{(n)}$, $V^{(n)}$, and $W^{(n)}$ (n = 1, 2, and 3) from formulas (3)–(5).

We will seek $U^{(n)}$, $V^{(n)}$, and $W^{(n)}$ in the form of the following series:

$$U^{(n)} = \sum_{m=0}^{\infty} \varepsilon^m U^{n,m} , \quad V^{(n)} = \sum_{m=0}^{\infty} \varepsilon^m V^{n,m} , \quad W^{(n)} = \sum_{m=0}^{\infty} \varepsilon^m W^{n,m} \quad (n = 1, 2, 3) . \tag{6}$$

The coefficients $U^{n,m}$, $V^{n,m}$, and $W^{n,m}$ are determined by substitution of the series (6) into Eqs. (2). A distinctive feature of system (2) is that the small parameter ε is involved in the coefficients for the second derivatives of the resolvents. On substituting (3) into (2), we use the transformations

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi_1} \frac{\partial \xi_1}{\partial x} = \varepsilon^{1/2} \frac{\partial}{\partial \xi_1} , \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi_1^2} \left(\frac{\partial \xi_1}{\partial x} \right)^2 = \varepsilon \frac{\partial^2}{\partial \xi_1^2} ; \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta_1} , \quad \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \eta_1^2} ,$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \zeta_1} , \quad \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \zeta_1^2} . \tag{7}$$

Analogously we transform the derivatives with respect to the variables (4) and (5). We introduce (3) and (7) into the first equation of system (2). As a result we obtain

$$\frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^2 u}{\partial y^2} + \varepsilon \frac{\partial^2 u}{\partial z^2} + e \frac{\partial^2 v}{\partial x \partial y} + e \frac{\partial^2 w}{\partial x \partial z} = \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^2 u}{\partial y^2} + \varepsilon \frac{\partial^2 u}{\partial z^2} + q\varepsilon \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) = 0 ,$$

where $q = (A_{12} + A_{44})/A_{44}$. Passing to the variables ξ_1, η_1, and ζ_1 in the last equation and reducing by ε, by virtue of (7) we have

$$\frac{\partial^2 U^{(1)}}{\partial \xi_1^2} + \frac{\partial^2 U^{(1)}}{\partial \eta_1^2} + \frac{\partial^2 U^{(1)}}{\partial \zeta_1^2} + q \left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right)$$

$$= \frac{\partial^2 U^{(1)}}{\partial \xi_1^2} + \frac{\partial^2 U^{(1)}}{\partial \eta_1^2} + \frac{\partial^2 U^{(1)}}{\partial \zeta_1^2} + q\varepsilon^{1/2}\varepsilon^{3/2}\left(\frac{\partial^2 V^{(1)}}{\partial \xi_1 \partial \eta_1} + \frac{\partial^2 W^{(1)}}{\partial \xi_1 \partial \zeta_1}\right) = 0$$

or

$$\frac{\partial^2 U^{(1)}}{\partial \xi_1^2} + \frac{\partial^2 U^{(1)}}{\partial \eta_1^2} + \frac{\partial^2 U^{(1)}}{\partial \zeta_1^2} + q\varepsilon^2\left(\frac{\partial^2 V^{(1)}}{\partial \xi_1 \partial \eta_1} + \frac{\partial^2 W^{(1)}}{\partial \xi_1 \partial \zeta_1}\right) = 0 . \tag{8}$$

This equation enables us to determine the expansion coefficients $U^{(1)}$ in the series of (6). The first two terms of the series of (6) represent harmonic functions, and the terms that follow are expanded in powers of the small parameter ε.

The last two equations of system (2), on substitution of the replacement (3) into them, are transformed to the form

$$\varepsilon^2 \frac{\partial^2 V^{(1)}}{\partial \xi_1^2} + \frac{\partial^2 V^{(1)}}{\partial \eta_1^2} + \varepsilon \frac{\partial^2 V^{(1)}}{\partial \zeta_1^2} + q \frac{\partial^2 U^{(1)}}{\partial \xi_1 \partial \eta_1} + q\varepsilon \frac{\partial^2 W^{(1)}}{\partial \eta_1 \partial \zeta_1} = 0 , \tag{9}$$

$$\varepsilon^2 \frac{\partial^2 W^{(1)}}{\partial \xi_1^2} + \varepsilon \frac{\partial^2 W^{(1)}}{\partial \eta_1^2} + \frac{\partial^2 W^{(1)}}{\partial \zeta_1^2} + q \frac{\partial^2 U^{(1)}}{\partial \xi_1 \partial \zeta_1} + q\varepsilon \frac{\partial^2 V^{(1)}}{\partial \eta_1 \partial \zeta_1} = 0 . \tag{10}$$

Equations (9) and (10) serve to determine $V^{(1)}$ and $W^{(1)}$ with the known $U^{(1)}$.

Analogously we transform the last two equations of system (2) on substitution of the replacement (4) into them. Omitting intermediate calculations, we give the equations for $V^{(2)}$ and $W^{(3)}$

$$\frac{\partial^2 V^{(2)}}{\partial \xi_2^2} + \frac{\partial^2 V^{(2)}}{\partial \eta_2^2} + \frac{\partial^2 V^{(2)}}{\partial \zeta_2^2} + q\varepsilon^2\left(\frac{\partial^2 U^{(2)}}{\partial \xi_2 \partial \eta_2} + \frac{\partial^2 W^{(2)}}{\partial \eta_2 \partial \zeta_2}\right) = 0 , \tag{11}$$

$$\frac{\partial^2 W^{(3)}}{\partial \xi_3^2} + \frac{\partial^2 W^{(3)}}{\partial \eta_3^2} + \frac{\partial^2 W^{(3)}}{\partial \zeta_3^2} + q\varepsilon^2\left(\frac{\partial^2 U^{(3)}}{\partial \xi_3 \partial \eta_3} + \frac{\partial^2 V^{(3)}}{\partial \xi_3 \partial \zeta_3}\right) = 0 . \tag{12}$$

The remaining equations are obtained analogously to what has been done for $V^{(1)}$ and $W^{(1)}$.

It follows from (8), (11), and (12) that $U^{n,m}$, $V^{n,m}$, and $W^{n,m}$ are harmonic functions at $n = \overline{1, 3}$ and $m =$ 0, 1. This enables us to use the theory of harmonic functions for finding the first two terms in the series (6) for $U^{(n)}$, $V^{(n)}$, and $W^{(n)}$, $n = \overline{1, 3}$. In determining the remaining terms of these series, we can use the method of small parameter. Since the resulting systems of equations have a complex form, following the known results of other authors [4], we can seek the components of the displacement vector as

$$U^{(n)} = \sum_{m=0}^{\infty} \sum_{j=0}^{1} \varepsilon^{m+\frac{j}{2}} U^{n,2m+j} , \quad V^{(n)} = \sum_{m=0}^{\infty} \sum_{j=0}^{1} \varepsilon^{m+\frac{j}{2}} V^{n,2m+j} , \quad W^{(n)} = \sum_{m=0}^{\infty} \sum_{j=0}^{1} \varepsilon^{m+\frac{j}{2}} W^{n,2m+j} \quad (n = \overline{1, 3}) . \tag{13}$$

Let us substitute the series (13) into Eqs. (8), (9), and (10); as a result we obtain

$$\sum_{m=0}^{\infty} \sum_{j=0}^{1} \varepsilon^{m+\frac{j}{2}} \Delta U^{1,2m+j} + q\varepsilon^2 \sum_{m=0}^{\infty} \sum_{j=0}^{1} \varepsilon^{m+\frac{j}{2}} \left(V_{\xi\eta}^{1,2m+j} + W_{\xi\zeta}^{1,2m+j}\right) = 0 ,$$

$$\sum_{m=0}^{\infty}\sum_{j=0}^{1}\varepsilon^{m+\frac{j}{2}}\left(\varepsilon^{2}V_{\xi\xi}^{1,2m+j}+V_{\eta\eta}^{1,2m+j}+\varepsilon V_{\zeta\zeta}^{1,2m+j}+qU_{\xi\eta}^{1,2m+j}+q\varepsilon W_{\eta\zeta}^{1,2m+j}\right)=0\,,$$

$$\sum_{m=0}^{\infty}\sum_{j=0}^{1}\varepsilon^{m+\frac{j}{2}}\left(\varepsilon^{2}W_{\xi\xi}^{1,2m+j}+\varepsilon W_{\eta\eta}^{1,2m+j}+W_{\zeta\zeta}^{1,2m+j}+qU_{\xi\zeta}^{1,2m+j}+q\varepsilon V_{\eta\zeta}^{1,2m+j}\right)=0\,.$$

Omitting intermediate calculations, we write the following systems of equations for $U^{1,n}$, $V^{1,n}$, and $W^{1,n}$, $n = \overline{1,4}$:

$$\varepsilon^{0}:\Delta U^{1,0}=0\,,\quad \varepsilon^{1/2}:\Delta U^{1,1}=0\,,\quad \varepsilon^{1}:\Delta U^{1,2}=0\,,\quad \varepsilon^{3/2}:\Delta U^{1,3}=0\,,$$

$$\varepsilon^{2}:\Delta U^{1,4}+q\left(V_{\xi\eta}^{1,0}+W_{\xi\zeta}^{1,0}\right)=0\,;$$

$$\varepsilon^{0}:V_{\eta\eta}^{1,0}+qU_{\xi\eta}^{1,0}=0\,,\quad \varepsilon^{1/2}:V_{\eta\eta}^{1,1}+qU_{\xi\eta}^{1,1}=0\,,\quad \varepsilon^{1}:V_{\eta\eta}^{1,2}+qU_{\xi\eta}^{1,2}+V_{\zeta\zeta}^{1,0}+qW_{\eta\zeta}^{1,0}=0\,,$$

$$\varepsilon^{3/2}:V_{\eta\eta}^{1,3}+qU_{\xi\eta}^{1,3}+V_{\zeta\zeta}^{1,1}+qW_{\eta\zeta}^{1,1}=0\,;$$

$$\varepsilon^{0}:W_{\zeta\zeta}^{1,0}+qU_{\xi\zeta}^{1,0}=0\,,\quad \varepsilon^{1/2}:W_{\zeta\zeta}^{1,1}+qU_{\xi\zeta}^{1,1}=0\,,\quad \varepsilon^{1}:W_{\zeta\zeta}^{1,2}+qU_{\xi\zeta}^{1,2}+W_{\eta\eta}^{1,0}+qV_{\eta\zeta}^{1,0}=0\,;$$

$$\varepsilon^{3/2}:W_{\zeta\zeta}^{1,3}+qU_{\xi\zeta}^{1,3}+W_{\eta\eta}^{1,1}+qV_{\eta\zeta}^{1,1}=0\,,\quad \varepsilon^{2}:W_{\zeta\zeta}^{1,4}+qU_{\xi\zeta}^{1,4}+W_{\eta\eta}^{1,2}+qV_{\eta\zeta}^{1,2}+W_{\xi\xi}^{1,0}=0\,.$$

Conclusions. We note that the systems of equations for $U^{(2)}$, $V^{(2)}$, $W^{(2)}$, $U^{(3)}$, $V^{(3)}$, and $W^{(3)}$ are transformed analogously. When the boundary problems for a cubically anisotropic medium are solved, it is necessary to satisfy boundary conditions that generally contain a parameter ε. If the boundary conditions contain no parameter ε, the solutions of the initial problem are much simplified, since the boundary conditions are used for finding the first approximation. The method of small parameter proposed in this work can also be extended to thermoelastic problems of the stressed-strained state of continua without substantial changes.

NOTATION

A_{ij}, material constants; A_{44}, shear strain; A_{11} and A_{12}, stretching-compression; u, v, and w, components of the displacement vector; ε, small parameter; ξ, η, and ζ, independent variables.

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